[11:00-11:20] Z-Transforms

The z-transform is related to the discrete-time Fourier transform, the discrete-time Fourier series, and the Laplace transform. It is used to describe the transfer of input to output frequencies of an LTI system, and converts convolution and difference equations into polynomials.

Recall that for an LTI system, the output is $y[n] = x[n] * h[n]$, the convolution of the input $x[n]$ with the system's impulse response $h[n]$.

The discrete-time Fourier transform of the output is $Y\left(e^{j\hat{\omega}}\right) = H\left(e^{j\hat{\omega}}\right)X\left(e^{j\hat{\omega}}\right)$, the product of the frequency response $H\left(e^{j\hat{\omega}}\right)$ with discrete-time Fourier transform if the input $X\left(e^{j\hat{\omega}}\right)$.

In the z-domain, we have a similar relation:

$$\underbrace{Y(z)}_{\substack{\text{z transform}\\ \text{of output}}} = \underbrace{H(z)}_{\substack{\text{transfer function}\\ \text{in z domain}}} \cdot \underbrace{X(z)}_{\substack{\text{z transform}\\ \text{of input}}}$$

When the substitution $z = e^{j\hat{\omega}}$ is valid, the frequency response is related to the transfer function in the z-domain

$$\underbrace{H\left(e^{j\hat{\omega}}\right)}_{\substack{\text{frequency}\\ \text{response}}} = \underbrace{H(z)}_{\substack{\text{transfer function}\\ \text{in z domain}}}\Bigg|_{z=e^{j\hat{\omega}}} \qquad \left(\text{when } z = e^{j\hat{\omega}} \text{ is valid}\right)$$

General definition (z is a complex variable).

$$\underbrace{x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]}_{\text{time domain}} \quad \overset{z}{\leftrightarrow} \quad \underbrace{X(z) = \sum_{k=-\infty}^{\infty} x[k]z^{-k}}_{\text{z domain}}$$

For an LTI system, the z transform $H(z)$ of the impulse response $h[n]$ is the system function or the transfer function in the z-domain.

Let $x[n] = z^n$ be the input to an LTI system with finite impulse response. The output is

$$y[n] = x[n] * h[n] = \sum_{k=0}^{M} h[k]x[n-k] = \sum_{k=0}^{M} h[k]z^{-k} = H(z)z^{-n}$$

[11:20-11:40] Examples of z transform pairs

The values z for which the z-transform is valid (not a divergent sum) is called the region of convergence (ROC)

Time domain: $x[n]$	z-domain: $X(z)$	ROC
Finite-length signal $x[n] = \sum_{k=0}^{N} x[k]\delta[n-k]$	Polynomial with powers of z $X(z) = \sum_{k=0}^{N} x[k]z^{-k}$	
Unit impulse $x[n] = \delta[n]$	Constant $X(z) = 1$	All z
Impulse delayed by n_0 samples $x[n] = \delta[n-n_0]$	z raised to the power of n_0 $X(z) = z^{-n_0}$	$z \neq 0$
Two-point impulse response $h[n] = h[0]\delta[n] + h[1]\delta[n-1]$	Polynomial in z^{-1} $H(z) = h[0] + h[1]z^{-1}$	$z \neq 0$
M-point FIR filter $\sum_{k=0}^{M} b_K\delta[n-k]$	Polynomial in z^{-1} $\sum_{k=0}^{M} b_k z^{-k}$	$z \neq 0$

Example of inverse z transform (find impulse response from transfer function)

If $H(z) = 1 + 2z^{-1} + z^{-2}$ what is $h[n]$?

$$h[n] = \delta[n] + 2\delta[n-1] + \delta[n-2]$$

[11:40-] Z transform properties

Time domain	Z domain
Superposition property	
$x[n] = ax_1[n] + bx_2[n]$	$X(z) = aX_1(z) + bX_2(z)$
Time delay property	
$y[n] = x[n-1]$	$Y(z) = x[-1] + z^{-1}X(z)$

[11:50] Convolution and the z-transform

Example: identity system. If an LTI system has impulse response $h[n] = \delta[n]$, then

$$y[n] = x[n] * \delta[n] = \underbrace{\sum_{k=-\infty}^{\infty} \delta[k]x[n-k]}_{\substack{\text{all terms other than} \\ k=0 \text{ are zero}}} = x[n]$$

In other words, the system copies the input to the output.

Example: Unit delay. $h[n] = \delta[n-1]$

$$y[n] = x[n] * h[n] = \underbrace{\sum_{k=-\infty}^{\infty} \delta[k-]x[n-k]}_{\substack{\text{all terms other than} \\ k=1 \text{ are zero}}} = x[n-1]$$

The system delays the input by one sample.

The transfer function in the z-domain is $H(z) = z^{-1}$.

The z-transform of the output is $Y(z) = H(z)X(z) = z^{-1}X(z)$.

In a block diagram, it's common to represent a delay by n samples as z^{-n}.

Convolution property

The output of an LTI system with finite impulse response is

$$y[n] = h[n] * x[n] = \underbrace{\sum_{k=0}^{M} h[k]x[n-k]}_{\substack{\text{assuming causal } x[n] \\ \text{i.e. } x[n]=0 \text{ for } n<0}}$$

The z-transform of the output is

$$Y(z) = \sum_{n=0}^{N} \left(\sum_{k=0}^{M} h[k]x[n-k] \right) z^{-n} = \sum_{k=0}^{M} h[k] \underbrace{\left(\sum_{n=0}^{N} x[n-k]z^{-n} \right)}_{\substack{z \text{ transform of x} \\ \text{delayed by k samples}}}$$

$$Y(z) = \sum_{k=0}^{M} h[k]\left(z^{-k}X(z)\right) = X(z)\sum_{k=0}^{M} h[k]z^{-k} = H(z)X(z)$$

Thus, convolution in the time domain becomes multiplication in the z domain.

$$y[n] = h[n] * x[n] \overset{z}{\leftrightarrow} Y(z) = H(z)X(z)$$

[12:10] Cascade of LTI systems

In the z-domain, the transfer function corresponding to cascading two LTI systems is the product of the transfer functions of each system:

$$y[n] = h_2[n] * (h_1[n] * x[n])$$

$$Y(z) = H_2(z)H_1(z)X(z)$$

This means that for strictly LTI systems, the order of operations for a cascade does not matter. In practice, the implementation of a system may break the LTI properties (e.g. finite numerical precision), causing the order of operations to matter.

[12:15] Relation between z domain and frequency domain

If $e^{j\hat{\omega}}$ is in the valid set of z values (region of convergence), the frequency response is

$$H\left(e^{j\hat{\omega}}\right) = H(z)\big|_{z=e^{j\hat{\omega}}}$$

10/30/25 Review LTI Systems

$$\underset{\delta[n]}{\xrightarrow{x[n]}} \boxed{LTI} \underset{h[n]}{\xrightarrow{y[n] = h[n] * x[n]}}$$

$X(e^{j\hat{\omega}})$ $\qquad\qquad$ $Y(e^{j\hat{\omega}}) = H(e^{j\hat{\omega}})X(e^{j\hat{\omega}})$

frequency response

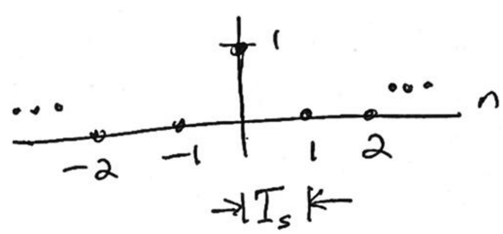
$$H(e^{j\hat{\omega}}) = |H(e^{j\hat{\omega}})| e^{j\measuredangle H(e^{j\hat{\omega}})}$$

$$X(z) \rightarrow \boxed{LTI} \rightarrow Y(z) = H(z)X(z)$$

a.k.a. transfer function in the freq. domain

transfer function in the z domain

$\delta[n]$ $\qquad\qquad$ $t = nT_s$



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$$h[n] = \delta[n] + 2\delta[n-1] + \delta[n-2]$$
$$H(z) = 1 + 2z^{-1} + z^{-2}$$

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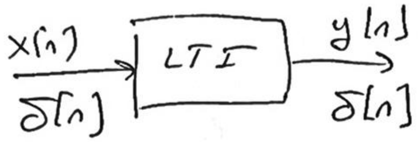
$y[n] = h[n] * x[n]$

Ex. #1



what system is this?

$h[n] = \delta[n]$

$y[n] = \delta[n] * x[n]$

$y[n] = \sum_{k=-\infty}^{\infty} \delta[k]\, x[n-k]$

$y[n] = x[n]$

Ex. #2

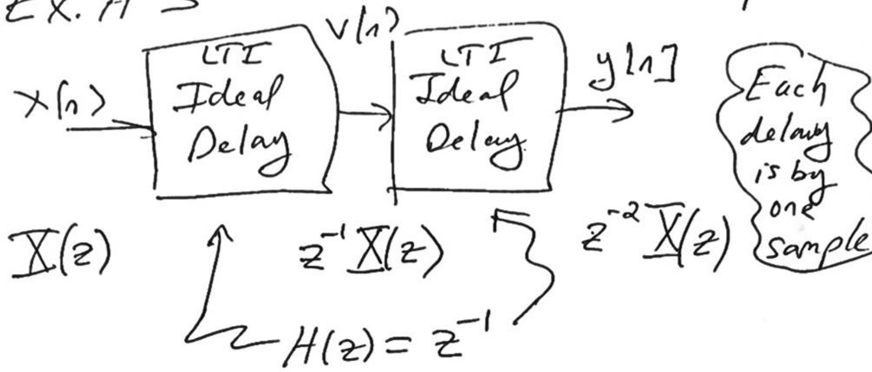
Unit Delay System

$y[n] = \delta[n-1] * x[n]$

$y[n] = \sum_{k=-\infty}^{\infty} \delta[k-1]\, x[n-k]$

$y[n] = x[n-1]$

Ex. #3 Slide 10-7



Each delay is by one sample

with K unit sample delays, output in z-domain, $z^{-K} X(z)$.